

# Counting indices of critical points of rank two of polynomial selfmaps of $\mathbb{R}^4$

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**Abstract** For a generic  $f \in C^\infty(\mathbb{R}^4, \mathbb{R}^4)$  there is a discrete set  $\Sigma^2(Df)$  of critical points of rank two, and there is an integer index  $I_p(Df)$  associated to any  $p \in \Sigma^2(Df)$ . We show how to compute  $\sum I_p(Df)$ ,  $p \in \Sigma^2(Df)$ , in the case where  $f$  is a polynomial mapping.

## 1 Introduction

Assume that  $f : M \rightarrow N$  is a  $C^1$ -mapping between 4-dimensional oriented manifolds. We shall denote by  $\Sigma^2(Df)$  the set of those critical points of  $f$  where the derivative  $Df$  is of rank two. According to the Thom transversality theorem, critical points of a generic  $f$  have rank  $\geq 2$ , and the set  $\Sigma^2(Df)$  is discrete.

Several authors observed that one may associate an index  $I_p(Df)$  to each  $p \in \Sigma^2(Df)$ . If  $M$  is a closed manifold then the algebraic sum of indices  $\#\Sigma^2(Df) = \sum_p I_p(Df)$ , where  $p \in \Sigma^2(Df)$ , is a natural invariant associated to  $f$  (see [2, 5, 11, 12, 13, 14], and [10] for the complex case). R. Stingley [13] proved that  $\#\Sigma^2(Df) = \deg(f) \cdot p_N - p_M$ , where  $\deg(f)$  is the topological degree of  $f$ , and  $p_N$  (resp.  $p_M$ ) is the Pontryagin number of  $N$  (resp.  $M$ ). This result demonstrates that there is a non-trivial linear relation between those two natural invariants, i.e.  $\#\Sigma^2(Df)$  and  $\deg(f)$ , associated to  $f$ .

In this paper we show how to compute  $\#\Sigma^2(Df)$  in the case where  $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is such a polynomial mapping that the family of critical points of rank two of its complexification is finite. Our approach does not exclude the case where points in  $\Sigma^2(Df)$  are not umbilic points.

Papers [5, 12, 13] offer methods of computing the index  $I_p(Df)$  which require the germ  $f : (\mathbb{R}^4, p) \rightarrow (\mathbb{R}^4, f(p))$  to be written in a special form. However, if  $f$  is a polynomial then we usually are not able to find explicitly points in  $\Sigma^2(Df)$ , so we cannot adopt these techniques in our case.

Let  $L$  denote the space of  $4 \times 4$ -matrices, and let  $\Sigma$  denote the connected oriented 12-dimensional submanifold of  $L$  consisting of matrices of rank two.

As  $Df : \mathbb{R}^4 \rightarrow L$ , one may define the index  $I_p(Df)$  as the intersection number of the mapping  $Df$  with  $\Sigma$  at  $Df(p) \in \Sigma$ . This is why in Sections 2,3 we investigate a more general case of mappings  $m : \mathbb{R}^4 \rightarrow L$ , and for  $p$  isolated in  $m^{-1}(\Sigma)$  we introduce the index  $I_p(m)$ .

In Section 4 we explain how to verify whether  $m_{\mathbb{R}}^{-1}(\Sigma)$  is finite in the case where  $m_{\mathbb{R}} : \mathbb{R}^4 \rightarrow L$  is a polynomial mapping. Then we construct a quadratic form whose signature equals  $\#\Sigma^2(m_{\mathbb{R}}) = \sum I_p(m_{\mathbb{R}})$ ,  $p \in m_{\mathbb{R}}^{-1}(\Sigma)$ . In the end of this section we present examples which were calculated with the help of SINGULAR [4]. We also give simple examples which demonstrate that there is only a trivial linear relation between the two natural integer invariants, i.e.  $\#\Sigma^2(Df)$  and  $\deg(f)$ , which one may associate to a polynomial  $f$ .

It is proper to add that [3, 8] present methods for counting the signed cusp or swallowtail singularities for polynomial selfmaps of  $\mathbb{R}^n$ , where  $n = 2, 3$ .

## 2 Permutations of rows or columns

In this section we show that a 3-cyclic permutation of either rows or columns of a  $4 \times 4$ -matrix may be prolonged to an isotopy preserving the rank of matrices.

Let  $W$  be a vector space, and let  $w_1, \dots, w_4 \in W$ . Put  $w_1(t) = (1-t)w_1 + tw_2$ ,  $w_2(t) = (1-t)w_2 + tw_3$ ,  $w_3(t) = (1-t)w_3 + tw_1$ ,  $w_4(t) = w_4$ . This way there is given an isotopy

$$\prod_{i=1}^4 W \times [0, 1] \ni ((w_i)_{i=1}^4, t) \mapsto (w_i(t))_{i=1}^4 \in \prod_{i=1}^4 W.$$

In particular  $(w_i(0))_{i=1}^4 = (w_1, w_2, w_3, w_4)$ , and  $(w_i(1))_{i=1}^4 = (w_2, w_3, w_1, w_4)$  is a 3-cyclic permutation of the first three vectors.

By the rank of a sequence of vectors we shall denote the dimension of the subspace spanned by these vectors.

**Lemma 2.1.**  $\text{rank}(w_i)_{i=1}^4 = 4$  if and only if  $\text{rank}(w_i(t))_{i=1}^4 = 4$  for all  $0 \leq t \leq 1$ .

*Proof.* As  $(1-t)^3 + t^3 > 0$  for all  $0 \leq t \leq 1$ , then the exterior product

$$\begin{aligned} w_1(t) \wedge w_2(t) \wedge w_3(t) \wedge w_4(t) &= (1-t)^3 w_1 \wedge w_2 \wedge w_3 \wedge w_4 + t^3 w_2 \wedge w_3 \wedge w_1 \wedge w_4 \\ &= ((1-t)^3 + t^3) w_1 \wedge w_2 \wedge w_3 \wedge w_4 \end{aligned}$$

does not vanish if and only if  $w_1 \wedge w_2 \wedge w_3 \wedge w_4 \neq 0$ .  $\square$

**Lemma 2.2.**  $\text{rank}(w_i)_{i=1}^4 \geq 3$  if and only if  $\text{rank}(w_i(t))_{i=1}^4 \geq 3$  for all  $0 \leq t \leq 1$ .

*Proof.* It is enough to prove  $(\Rightarrow)$ . If  $w_1, w_2, w_3$  are linearly independent then  $w_1 \wedge w_2 \wedge w_3 \neq 0$ , and so

$$w_1(t) \wedge w_2(t) \wedge w_3(t) = ((1-t)^3 + t^3)w_1 \wedge w_2 \wedge w_3 \neq 0$$

for all  $0 \leq t \leq 1$ .

If  $w_1, w_2, w_3$  are linearly dependent, then at least one of the products  $w_1 \wedge w_2 \wedge w_4$ ,  $w_1 \wedge w_3 \wedge w_4$ ,  $w_2 \wedge w_3 \wedge w_4$  is non-zero. We have

$$\begin{bmatrix} w_1(t) \wedge w_2(t) \wedge w_4 \\ w_1(t) \wedge w_3(t) \wedge w_4 \\ w_2(t) \wedge w_3(t) \wedge w_4 \end{bmatrix} = \begin{bmatrix} (1-t)^2 & t(1-t) & t^2 \\ -t^2 & (1-t)^2 & t(1-t) \\ -t(1-t) & -t^2 & (1-t)^2 \end{bmatrix} \begin{bmatrix} w_1 \wedge w_2 \wedge w_4 \\ w_1 \wedge w_3 \wedge w_4 \\ w_2 \wedge w_3 \wedge w_4 \end{bmatrix}.$$

The determinant of the above matrix equals  $((1-t)^3 + t^3)^2$ , so it does not vanish in  $[0, 1]$ . Hence, for any  $0 \leq t \leq 1$ , at least one of the products  $w_i(t) \wedge w_j(t) \wedge w_4$  does not vanish, and then  $\text{rank}(w_i(t))_{i=1}^4 \geq 3$ .  $\square$

**Lemma 2.3.**  $\text{rank}(w_i)_{i=1}^4 \geq 2$  if and only if  $\text{rank}(w_i(t))_{i=1}^4 \geq 2$  for all  $0 \leq t \leq 1$ .

*Proof.* Assume first that  $\text{rank}(w_1, w_2, w_3) \geq 2$ . Then at least one of the products  $w_1 \wedge w_2$ ,  $w_1 \wedge w_3$ ,  $w_2 \wedge w_3$  is non-zero. We have

$$\begin{bmatrix} w_1(t) \wedge w_2(t) \\ w_1(t) \wedge w_3(t) \\ w_2(t) \wedge w_3(t) \end{bmatrix} = \begin{bmatrix} (1-t)^2 & t(1-t) & t^2 \\ -t^2 & (1-t)^2 & t(1-t) \\ -t(1-t) & -t^2 & (1-t)^2 \end{bmatrix} \begin{bmatrix} w_1 \wedge w_2 \\ w_1 \wedge w_3 \\ w_2 \wedge w_3 \end{bmatrix}.$$

By the same arguments as in Lemma 2.2,  $\text{rank}(w_1(t), w_2(t), w_3(t)) \geq 2$  for all  $0 \leq t \leq 1$ .

If  $\text{rank}(w_1, w_2, w_3) \leq 1$ , then at least one of the products  $w_1 \wedge w_4$ ,  $w_2 \wedge w_4$ ,  $w_3 \wedge w_4$  is non-zero. We have

$$\begin{bmatrix} w_1(t) \wedge w_4 \\ w_2(t) \wedge w_4 \\ w_3(t) \wedge w_4 \end{bmatrix} = \begin{bmatrix} 1-t & t & 0 \\ 0 & 1-t & t \\ t & 0 & 1-t \end{bmatrix} \begin{bmatrix} w_1 \wedge w_4 \\ w_2 \wedge w_4 \\ w_3 \wedge w_4 \end{bmatrix}.$$

The determinant of the above matrix equals  $(1-t)^3 + t^3$ . Hence for any  $0 \leq t \leq 1$  at least one of the products  $w_i(t) \wedge w_4$  does not vanish, and then  $\text{rank}(w_i(t))_{i=1}^4 \geq 2$ .  $\square$

**Lemma 2.4.**  $\text{rank}(w_i)_{i=1}^4 \geq 1$  if and only if  $\text{rank}(w_i(t))_{i=1}^4 \geq 1$  for all  $0 \leq t \leq 1$ .

*Proof.* If  $w_4 \neq 0$  then the assertion is obvious. Suppose that at least one  $w_i \neq 0$ , where  $1 \leq i \leq 3$ . Then

$$\begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} = \begin{bmatrix} 1-t & t & 0 \\ 0 & 1-t & t \\ t & 0 & 1-t \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix},$$

and we may apply the same arguments as in the previous lemma.  $\square$

**Corollary 2.5.** *Obviously, if  $\text{rank}(w_i(t))_{i=1}^4 \geq k$  for at least one  $t$ , then  $\text{rank}(w_i)_{i=1}^4 \geq k$ . Hence  $\text{rank}(w_i)_{i=1}^4 = k$  if and only if  $\text{rank}(w_i(t))_{i=1}^4 = k$  for each  $0 \leq t \leq 1$ .*

Let  $L$  denote the linear space of  $4 \times 4$ -matrices

$$M = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with real coordinates  $(a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, \dots, c_{22}, d_{11}, d_{12}, d_{21}, d_{22})$ .

There is a natural isomorphism  $L \simeq \prod_{i=1}^4 \mathbb{R}^4 = \{(w_i)_{i=1}^4 \mid w_i \in \mathbb{R}^4\}$ , where the sequence  $(w_i)_{i=1}^4$  denotes either rows or columns of  $M$ . We get immediately

**Corollary 2.6.** *Let  $\tau : L \rightarrow L$  be a composition of finite sequence of 3-cyclic permutations of either rows or columns. (It is worth to notice that a permutation consisting of two disjoint transpositions is a composition of a finite sequence of 3-cyclic permutations.) Then there exists an isotopy  $T : L \times [0, 1] \rightarrow L$  such that  $T(M, 0) \equiv M$ ,  $T(M, 1) = \tau(M)$ , and  $\text{rank } T(M, t) \equiv \text{rank } M$ .*

### 3 Index of a critical point

In this section we investigate the intersection number of a germ  $(\mathbb{R}^4, p) \rightarrow L$  with the submanifold consisting of matrices of rank two, and then we introduce the index of a critical point of rank two.

Denote by  $U_{\pm}$  the open connected subsets of  $L$  consisting of matrices with  $\pm \det(A) > 0$ , and let  $U = U_+ \cup U_-$ . Put  $\Sigma = \{M \in L \mid \text{rank}(M) = 2\}$ . By [6, Proposition 2.5.3], a matrix  $M \in U$  has rank two if and only if  $D - CA^{-1}B = 0$ . Then  $\Sigma \cap U$  is the graph of the function  $D = CA^{-1}B$ , so that  $\dim \Sigma \cap U = 12$ ,  $\text{codim } \Sigma \cap U = 4$ . There is also given the natural orientation of the normal bundle over  $\Sigma \cap U_+$  induced by the orientation of coordinates  $(d_{11}, d_{12}, d_{21}, d_{22})$  in the source of this function.

Let  $M_{ij}$ , where  $3 \leq i, j \leq 4$ , denote the minor obtained by removing the  $i$ -th row and the  $j$ -th column from  $M$ . One may check that

$$\det(A) \cdot (D - CA^{-1}B) = \begin{bmatrix} M_{44} & M_{43} \\ M_{34} & M_{33} \end{bmatrix},$$

so that  $\Sigma \cap U = \bigcap M_{ij}^{-1}(0) \cap U$ . Moreover

$$\frac{\partial(M_{44}, M_{43}, M_{34}, M_{33})}{\partial(d_{11}, d_{12}, d_{21}, d_{22})} = (\det(A))^4$$

is positive on  $U$ . In the further part of this paper we will need

**Lemma 3.1.** *Let  $M'$  denote the matrix obtained by interchanging the first row of  $M \in U_+$  with the second one, and the third row with the fourth one, so that  $M' \in U_-$ . Let  $M'_{ij}$  denote the minor obtained by removing the  $i$ -th row and the  $j$ -th column from  $M'$ . Then*

$$\begin{bmatrix} M'_{44} \\ M'_{43} \\ M'_{34} \\ M'_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_{44} \\ M_{43} \\ M_{34} \\ M_{33} \end{bmatrix},$$

and the determinant of the above  $4 \times 4$ -matrix equals  $+1$ .  $\square$

**Lemma 3.2.** *Let  $\mathcal{R}$  denote the localization of the ring of polynomials on  $L$  by the powers of  $\det(A)$ . Then the ideal in  $\mathcal{R}$  generated by all  $3 \times 3$ -minors of  $M$  equals by the one generated by  $M_{ij}$ , where  $3 \leq i, j \leq 4$ .*

*Proof.* Applying elementary operations on rows and collumns, one may transform the matrix  $M$  to the form

$$M'' = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & M_{44}/\det(A) & M_{43}/\det(A) \\ 0 & 0 & M_{34}/\det(A) & M_{33}/\det(A) \end{bmatrix}.$$

By the Cauchy-Binet formula, the ideal in  $\mathcal{R}$  generated by all  $3 \times 3$ -minors of  $M$  is equal to the one generated by all  $3 \times 3$ -minors of  $M''$ , i.e. by all  $M_{ij}$  and all  $a_{ij} \cdot (M_{33}M_{44} - M_{34}M_{43})/\det(A)^2$ . As  $\det(A)$  is invertible in  $\mathcal{R}$ , the last ideal is generated by all  $M_{ij}$ .  $\square$

It is well-known that  $\Sigma$  is a connected submanifold of  $L$  of codimension 4. According to [1, Proposition 4.1], the manifold  $\Sigma$ , as well as its normal bundle, is orientable. Let fix the global orientation of of the normal bundle over  $\Sigma$  which coincides with the orientation of the normal bundle over  $\Sigma \cap U_+$  introduced before.

**Definition.** Let  $V$  be an open neighbourhood of  $p \in \mathbb{R}^4$ , and let  $m : V \rightarrow L$  be a continuous mapping such that  $p$  is isolated in  $m^{-1}(\Sigma)$ . We define the index  $I_p(m)$  as the intersection number of  $m$  with  $\Sigma$  at  $m(p)$ . In particular, if  $m(p) \in \Sigma \cap U_+$  then  $I_p(m)$  is the local topological degree of the mapping  $(\mathbb{R}^4, p) \ni x \mapsto H(x) = (h_{44}(x), h_{43}(x), h_{34}(x), h_{33}(x)) \in (\mathbb{R}^4, \mathbf{0})$ , where  $h_{ij}(x) = M_{ij}(m(x))$ . By Corollary 2.6 and Lemma 3.1, if  $m(p) \in \Sigma \cap U_-$ , then  $I_p(m)$  equals the local topological degree of the same mapping. If  $p = \mathbf{0}$  is the origin in  $\mathbb{R}^4$ , we shall denote its index by  $I(m)$ .

In the remainder of this section and in the next one we shall assume that  $p = \mathbf{0}$  is isolated in  $m^{-1}(\Sigma)$ .

Of course, if there is a continuous family of mappings  $m_t : V \rightarrow L$ , where  $t \in [0, 1]$ , such that  $m_t(x) \in \Sigma$  if and only if  $x = \mathbf{0}$ , then  $I(m_0) = I(m_1)$ .

**Proposition 3.3.** *Assume that  $\tau : L \rightarrow L$  is a composition of a finite sequence of 3-cyclic permutations of rows or columns such that  $\tau(m(\mathbf{0})) \in \Sigma \cap U$ . Then  $I(m) = I(\tau \circ m)$ .*

*Proof.* We may assume that  $m^{-1}(\Sigma) \cap V = \{\mathbf{0}\}$ . Take such an isotopy  $T : L \times [0, 1] \rightarrow L$  as in Corollary 2.6. Put  $m_t(x) = T(m(x), t)$ . Then  $m_0 = T(m, 0) = m$ ,  $m_1 = T(m, 1) = \tau \circ m$ , so that  $m_1(\mathbf{0}) \in \Sigma \cap U$ . Moreover  $m_t(x) \in \Sigma$  if and only if  $x = \mathbf{0}$ . Hence  $I(m) = I(m_1) = I(\tau \circ m)$ .  $\square$

**Example.** Let

$$m(x, y, z, w) = \begin{bmatrix} x & y & z & 0 \\ z^3 & w & 0 & 0 \\ 0 & 0 & 1-x & y \\ w & 0 & 0 & 1-z \end{bmatrix}.$$

Then  $m(\mathbf{0}) \in \Sigma \setminus U$ . After applying a finite sequence  $\tau$  of 3-cyclic permutations of rows or columns we get

$$\tau \circ m(x, y, z, w) = \begin{bmatrix} 1-x & y & 0 & 0 \\ 0 & 1-z & w & 0 \\ z & 0 & x & y \\ 0 & 0 & z^3 & w \end{bmatrix},$$

so that  $\tau(m(\mathbf{0})) \in \Sigma \cap U_+$ . Then

$$\begin{aligned} h_{44} &= \begin{bmatrix} 1-x & y & 0 \\ 0 & 1-z & w \\ z & 0 & x \end{bmatrix} = (1-x)(1-z)x + yzw, \\ h_{43} &= \begin{bmatrix} 1-x & y & 0 \\ 0 & 1-z & 0 \\ z & 0 & y \end{bmatrix} = (1-x)(1-z)y, \\ h_{34} &= \begin{bmatrix} 1-x & y & 0 \\ 0 & 1-z & w \\ 0 & 0 & z^3 \end{bmatrix} = (1-x)(1-z)z^3, \\ h_{33} &= \begin{bmatrix} 1-x & y & 0 \\ 0 & 1-z & 0 \\ 0 & 0 & w \end{bmatrix} = (1-x)(1-z)w. \end{aligned}$$

Since the local topological degree  $\deg_0(H) = +1$ , we have  $I(m) = +1$ .

## 4 Polynomial mappings

Let  $\mathbb{K}$  denote either  $\mathbb{R}$  or  $\mathbb{C}$ . For  $p \in \mathbb{K}^4$ , let  $\mathcal{O}_{\mathbb{K},p}$  denote the ring of germs at  $p$  of analytic functions  $(\mathbb{K}^4, p) \rightarrow \mathbb{K}$ , and let  $\mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_4]$ . Let  $L_{\mathbb{C}}$  denote the space of  $4 \times 4$ -matrices having complex entries, and let  $\Sigma_{\mathbb{C}} = \{M \in L_{\mathbb{C}} \mid \text{rank}(M) = 2\}$ .

**Lemma 4.1.** *Assume that matrices  $M_1, \dots, M_s$  belong to  $\Sigma_{\mathbb{C}}$ . There exists an open dense subset  $\Delta \subset L \times L$  such that for any  $(L_1, L_2) \in \Delta$  and each  $1 \leq i \leq s$ , the leading principal minor of  $L_1 \cdot M_i \cdot L_2$  of order 2 does not vanish.*

*Proof.* For  $M \in L_{\mathbb{C}}$ , let  $a(M)$  denote its leading principal minor of order 2. Then  $a : L_{\mathbb{C}} \rightarrow \mathbb{C}$  is a polynomial mapping.

Each matrix  $M_i$  is of rank 2, so that at least one of its  $2 \times 2$ -minors does not vanish. There exists  $(L_1^i, L_2^i) \in L \times L$ , which represents the appropriate interchange of rows and columns, such that  $a(L_1^i \cdot M_i \cdot L_2^i) \neq 0$ . Then  $\Delta_i = \{(L_1, L_2) \in L \times L \mid a(L_1 \cdot M_i \cdot L_2) \neq 0\}$  is a non-empty Zarisky open subset of  $L \times L$ . The set  $\Delta = \bigcap_1^s \Delta_i$  satisfies the assertion.  $\square$

Let

$$m_{\mathbb{R}}(x) = \begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix} : \mathbb{R}^4 \rightarrow L$$

be a polynomial mapping, and let  $m_{\mathbb{C}} : \mathbb{C}^4 \rightarrow L_{\mathbb{C}}$  be its complexification. Let  $H_{\mathbb{K}} = (h_{44}, h_{43}, h_{34}, h_{33}) : \mathbb{K}^4 \rightarrow \mathbb{K}^4$ , where  $h_{ij}(x) = M_{ij}(m_{\mathbb{K}}(x))$ .

Denote by  $S_{\mathbb{K}}$  the ideal in  $\mathbb{K}[x]$  generated by all  $3 \times 3$ -minors of  $m_{\mathbb{K}}(x)$ . Set  $\mathcal{A}_{\mathbb{K}} = \mathbb{K}[x]/S_{\mathbb{K}}$ . For  $p \in V(S_{\mathbb{K}})$ , denote  $\mathcal{A}_{\mathbb{K},p} = \mathcal{O}_{\mathbb{K},p}/S_{\mathbb{K}}$ . From now on we shall assume that  $\dim_{\mathbb{R}} \mathcal{A}_{\mathbb{R}} = \dim_{\mathbb{C}} \mathcal{A}_{\mathbb{C}} < \infty$ . Then the set  $V(S_{\mathbb{K}})$  of zeros of  $S_{\mathbb{K}}$  in  $\mathbb{K}^4$ , which equals  $\{p \in \mathbb{K}^4 \mid \text{rank}(m_{\mathbb{K}}(p)) \leq 2\}$ , is finite.

Let  $P_{\mathbb{K}} \subset \mathbb{K}[x]$  denote the ideal generated by all  $2 \times 2$ -minors of  $m_{\mathbb{K}}(x)$ . From now on we shall assume that  $P_{\mathbb{K}} = \mathbb{K}[x]$ , so that  $\text{rank}(m_{\mathbb{K}}(p)) \geq 2$  at any  $p \in \mathbb{K}^4$  and then  $V(S_{\mathbb{K}}) = \{p \in \mathbb{K}^4 \mid \text{rank}(m_{\mathbb{K}}(p)) = 2\}$ .

In particular, if  $p \in V(S_{\mathbb{R}})$  (resp.  $p \in V(S_{\mathbb{C}})$ ) then  $p$  is isolated in  $m_{\mathbb{R}}^{-1}(\Sigma)$  (resp. in  $m_{\mathbb{C}}^{-1}(\Sigma_{\mathbb{C}})$ ). Hence, for  $p \in V(S_{\mathbb{R}})$  the intersection index  $I_p(m_{\mathbb{R}})$  is defined, and is equal to the local topological degree  $\deg_p(H_{\mathbb{R}}) : (\mathbb{R}^4, p) \rightarrow (\mathbb{R}^4, \mathbf{0})$ .

**Definition.** Set  $\#\Sigma^2(m_{\mathbb{R}}) = \sum I_p(m_{\mathbb{R}})$ , where  $p \in V(S_{\mathbb{R}}) = m_{\mathbb{R}}^{-1}(\Sigma)$ .

Be Lemma 4.1, after  $\mathbb{R}$ -linear changes of coordinates in  $\mathbb{C}^n$  if necessary, one may expect  $\det(A)$  not to vanish at every  $p \in V(S_{\mathbb{K}})$ , so that the ideal in  $\mathbb{K}[x]$  generated by  $S_{\mathbb{K}}$  and  $\det(A)$  equals  $\mathbb{K}[x]$ . This justifies assumptions of the next lemma.

**Lemma 4.2.** *Assume that the ideal generated by  $S_{\mathbb{K}}$  and  $\det(A)$  equals  $\mathbb{K}[x]$ . Then*

- (i)  $\det(A(p)) \neq 0$  at each  $p \in V(S_{\mathbb{K}})$ ,



(ii) at each  $p \in V(S_{\mathbb{K}})$ , the ideal in  $\mathcal{O}_{\mathbb{K},p}$  generated by  $S_{\mathbb{K}}$  equals the one generated by  $h_{44}, h_{43}, h_{34}, h_{33}$ .

*Proof.* Assertion (i) is obvious. If  $p \in V(S_{\mathbb{K}})$  then  $\det(A(p)) \neq 0$ . By Lemma 3.2, the ideal in  $\mathcal{O}_{\mathbb{K},p}$  generated by  $S_{\mathbb{K}}$ , i.e. by all  $3 \times 3$ -minors of  $m_{\mathbb{K}}(x)$ , is in fact generated by  $M_{ij}(m_{\mathbb{K}}(x)) = h_{ij}(x)$ , where  $3 \leq i, j \leq 4$ .  $\square$

Assertion (ii) of the above lemma allows us to compute  $\#\Sigma^2(m_{\mathbb{R}})$  by applying arguments developed in [7, Section 3]. We shall now recall briefly the method presented there.

Let  $V(S_{\mathbb{C}}) = \{p_1, \dots, p_r\}$ . The complex conjugation on  $V(S_{\mathbb{C}})$  fixes  $V(S_{\mathbb{R}})$ , so one may assume that  $V(S_{\mathbb{R}}) = \{p_1, \dots, p_m\}$  and  $V(S_{\mathbb{C}}) \setminus V(S_{\mathbb{R}})$  is the union of pairs of conjugate points  $\{p_{m+1}, \overline{p_{m+1}}, \dots, p_w, \overline{p_w}\}$ , where  $w = (r - m)/2$ . Put  $h_1 = h_{44}, h_2 = h_{43}, h_3 = h_{34}, h_4 = h_{33}$ .

For  $x = (x_1, \dots, x_4)$ ,  $x' = (x'_1, \dots, x'_4)$ , and  $1 \leq i, j \leq 4$  define

$$T_{ij}(x, x') = \frac{h_i(x'_1, \dots, x_j, \dots, x_4) - h_i(x'_1, \dots, x'_j, \dots, x_4)}{x_j - x'_j}.$$

It is easy to see that each  $T_{ij}$  extends to a polynomial, thus we may assume that  $T_{ij} \in \mathbb{R}[x, x'] = \mathbb{R}[x_1, \dots, x_4, x'_1, \dots, x'_4]$ . There is the natural projection  $\mathbb{R}[x, x'] \longrightarrow \mathcal{A}_{\mathbb{R}} \otimes \mathcal{A}_{\mathbb{R}}$  given by

$$x_1^{\alpha_1} \cdots x_4^{\alpha_4} (x'_1)^{\beta_1} \cdots (x'_4)^{\beta_4} \mapsto x_1^{\alpha_1} \cdots x_4^{\alpha_4} \otimes (x'_1)^{\beta_1} \cdots (x'_4)^{\beta_4}.$$

Let  $T$  denote the image of  $\det[T_{ij}(x, x')]$  in  $\mathcal{A}_{\mathbb{R}} \otimes \mathcal{A}_{\mathbb{R}}$ .

Put  $d = \dim_{\mathbb{R}} \mathcal{A}_{\mathbb{R}}$ . Assume that  $e_1, \dots, e_d$  form a basis in  $\mathcal{A}_{\mathbb{R}}$ . So  $\dim_{\mathbb{R}} \mathcal{A}_{\mathbb{R}} \otimes \mathcal{A}_{\mathbb{R}} = d^2$  and  $e_i \otimes e_j$ , for  $1 \leq i, j \leq d$ ,

form a basis in  $\mathcal{A}_{\mathbb{R}} \otimes \mathcal{A}_{\mathbb{R}}$ . Hence there are  $t_{ij} \in \mathbb{R}$  such that

$$T = \sum_{i,j=1}^d t_{ij} e_i \otimes e_j = \sum_{i=1}^d e_i \otimes \hat{e}_i,$$

where  $\hat{e}_i = \sum_{j=1}^d t_{ij} e_j$ . Elements  $\hat{e}_1, \dots, \hat{e}_d$  form a basis in  $\mathcal{A}_{\mathbb{R}}$ . So there are  $A_1, \dots, A_d \in \mathbb{R}$  such that  $1 = A_1 \hat{e}_1 + \dots + A_d \hat{e}_d$  in  $\mathcal{A}_{\mathbb{R}}$ .

**Definition.** For  $a = a_1 e_1 + \dots + a_d e_d \in \mathcal{A}_{\mathbb{R}}$  define  $\varphi_T(a) = a_1 A_1 + \dots + a_d A_d$ . Hence  $\varphi_T : \mathcal{A}_{\mathbb{R}} \longrightarrow \mathbb{R}$  is a linear functional. Let  $\Phi_T$  be the bilinear form on  $\mathcal{A}_{\mathbb{R}}$  given by  $\Phi_T(a, b) = \varphi_T(ab)$ .

**Theorem 4.3.** [7, Theorem 14, p. 275] The form  $\Phi_T$  is non-degenerate and

$$\sum_{i=1}^m \deg_{p_i}(H_{\mathbb{R}}) = \text{signature } \Phi_T. \quad \square$$

**Theorem 4.4.** Suppose that  $\dim_{\mathbb{R}} \mathcal{A}_{\mathbb{R}} < \infty$  and the ideal generated by  $S_{\mathbb{R}}$  and  $\det(A)$  equals  $\mathbb{R}[x]$ . Then

$$\#\Sigma^2(m_{\mathbb{R}}) = \text{signature}(\Phi_T). \quad \square$$

*Proof.* It is enough to observe that  $\#\Sigma^2(m_{\mathbb{R}}) = \sum_{i=1}^m I_{p_i}(m_{\mathbb{R}}) = \sum_{i=1}^m \deg_{p_i}(H_{\mathbb{R}})$ .  $\square$

**Example.** Let  $f = (x - 2y^2 + zw, y - x^2w + 4z^3, zw + 3w + x^2, xz + yw - 4y) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ , and  $m_{\mathbb{R}} = Df$ . Using SINGULAR [4] and computer programs written by Adriana Gorzelak and Magdalena Sarnowska - students of computer sciences at the Gdańsk University - one may verify that  $\dim_{\mathbb{R}} \mathcal{A}_{\mathbb{R}} = 34$ , other assumptions of the above theorem hold, and  $\text{signature}(\Phi_T) = 2$ , so that  $\#\Sigma^2(Df) = 2$ .

**Example.** Let  $f = (x - z^3, y - xzw, x^3 - yz + yw, x^2 + y^2 + zw) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ , and  $m_{\mathbb{R}} = Df$ . One may verify that  $\dim_{\mathbb{R}} \mathcal{A}_{\mathbb{R}} = 23$ , other assumptions of the above theorem hold, and  $\text{signature}(\Phi_T) = 1$ , so that  $\#\Sigma^2(Df) = 1$ . Moreover  $\text{rank}(Df(\mathbf{0})) = 2$ ,  $I_0(Df) = -1$  and  $\dim_{\mathbb{R}} \mathcal{A}_{\mathbb{R}, \mathbf{0}} = 3$ , so that the origin is not an umbilic point.

If  $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is a proper polynomial mapping, there exists  $R > 0$  such that  $|x| < R$  for each  $x \in f^{-1}(\mathbf{0})$ . Denote by  $\deg(f)$  the topological degree of  $S_R^3 \ni x \mapsto f(x)/|f(x)| \in S^3$ .

**Example.** Let  $f_{\pm} = (x, y, z^2 - w^2 \pm xz + yw, \pm zw)$ . Both  $f_{\pm}$  are proper and  $(Df_{\pm})^{-1}(\Sigma) = f_{\pm}^{-1}(\mathbf{0}) = \{\mathbf{0}\}$ . We have

$$Df_{\pm} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pm z & w & 2z \pm x & -2w + y \\ 0 & 0 & \pm w & \pm z \end{bmatrix},$$

so that  $Df_{\pm}(\mathbf{0}) \in \Sigma \cap U$ . Then  $\#\Sigma^2(Df) = I_{\mathbf{0}}(Df_{\pm}) = \deg_{\mathbf{0}}(2z \pm x, -2w + y, \pm w, \pm z) = \mp 1$ . Moreover,  $\deg(f_{\pm}) = \deg_{\mathbf{0}}(f_{\pm}) = \pm 2$ .

**Example.** Let  $g_{\pm} = (x, y, z^2 + w^2 \pm xz + yw, \pm zw)$ . Both  $g_{\pm}$  are proper and  $(Dg_{\pm})^{-1}(\Sigma) = g_{\pm}^{-1}(\mathbf{0}) = \{\mathbf{0}\}$ . We have

$$Dg_{\pm} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pm z & w & 2z \pm x & 2w + y \\ 0 & 0 & \pm w & \pm z \end{bmatrix},$$

so that  $Dg_{\pm}(\mathbf{0}) \in \Sigma \cap U$ . Then  $\#\Sigma^2(Dg) = I_{\mathbf{0}}(Dg_{\pm}) = \deg_{\mathbf{0}}(2z \pm x, 2w + y, \pm w, \pm z) = \mp 1$ . Moreover,  $\deg(g_{\pm}) = \deg_{\mathbf{0}}(g_{\pm}) = 0$ .

The above examples demonstrate that there is only a trivial linear relation of the form  $A\#\Sigma^2(Df) + B\deg(f) + C = 0$  for mappings  $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ , contrary to the case of mappings  $M \rightarrow N$ , where  $M$  is closed (see [13]).

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